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Krein's resolvent formula for self-adjoint extensions of symmetric second-order elliptic differential operators

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Abstract

Given a symmetric, semi-bounded, second-order elliptic differential operator A on a bounded domain with $C^{1,1}$ boundary, we provide a Kreĭn-type formula for the resolvent difference between its Friedrichs extension and an arbitrary self-adjoint one.

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1. Introduction

Given a bounded open set $\Omega \subset \mathbb{R}^n$, $n > 1$, let us consider a second-order elliptic differential operator

$$A : C_c^\infty(\Omega) \subset L^2(\Omega) \to L^2(\Omega), \qquad A = \sum_{i,j=1}^{n} \partial_i(a_{ij}\partial_j) - \sum_{i=1}^{n} b_i\partial_i - c.$$

Such an operator A, under appropriate hypotheses on its coefficients and on Ω (these will be made precise in section 3), is closable and its closure $A_{\min}$, the minimal realization of A, has a domain given by $H_0^2(\Omega)$, the closure of $C_c^\infty(\Omega)$ with respect to the $H^2(\Omega)$ Sobolev norm. If A is symmetric then $A_{\min}$ is symmetric but not self-adjoint, i.e. A is not essentially self-adjoint. Indeed $A_{\min}^* = A_{\max}$, where $A_{\max}$ the maximal realization of A, has the domain made by the functions $u \in L^2(\Omega)$ such that $Au \in L^2(\Omega)$. Assuming that $A_{\min}$ is semi-bounded, then $A_{\min}$ has a self-adjoint extension A_0 (the Friedrichs extension, corresponding to Dirichlet boundary conditions), $A_{\min} \subsetneq A_0 \subsetneq A_{\max}$, and hence $A_{\min}$ has infinitely many self-adjoint extensions.

The problem of the parametrization of all self-adjoint extensions of $A_{\min}$ in terms of boundary conditions was completely solved (in the case of an elliptic differential operator of arbitrary order) in [12] (for some older papers about similar topics we just quote [5, 23]). Here, by using the approach developed in [16–19], we give an alternative derivation of such a result by providing a Kreĭn-like formula for the resolvent difference between an arbitrary self-adjoint extension of $A_{\min}$ and its Friedrichs extension A_0. For the sake of simplicity here we consider the case of a second-order differential operator. The case of higher order operators can be treated in a similar way.

In the case A is the Laplacian, the Kreĭn resolvent formula presented here has been given in [19], example 5.5. For other recent results on the Kreĭn-type formula for partial differential operators see [1, 3, 4, 8, 9, 22].

In order to help the reader's intuition on the results presented here, in section 4 we consider one of the simplest possible examples: a rotation-invariant elliptic operators A on the disc $D \subset \mathbb{R}^2$. Thus, notwithstanding the symmetric operator considered here has infinite deficiency indices, due to the presence of symmetries the resolvents of their self-adjoint extensions can be written, by separation of variables, in a form which resembles the finite indices case (see the comments in remark 4.1), and the corresponding spectral analysis becomes simpler. As illustrated, given any sequence $\{\lambda_n\}_1^\infty \subset \mathbb{R}$, boundary conditions at ∂D can be given for which A is self-adjoint and such that $\{\lambda_n\}_1^\infty$ is contained in its point spectrum. Remark 4.3 shows that such boundary conditions can be quite different from the usual ones.

2. Preliminaires

For the reader's convenience in this section we collect some results from [16–19]. We refer to these papers, in particular to [19], for a thorough discussion about the connection of the approach presented here with both the standard von Neumann's theory of self-adjoint extension [15] and with boundary triple theory [6, 10].

From now on we will denote by

$$\mathscr{D}(L), \quad \mathscr{K}(L), \quad \mathscr{R}(L), \quad \rho(L)$$

the domain, kernel, range and resolvent set of a linear operator L.

Let $\mathscr{H}$ a Hilbert space with scalar product $\langle\cdot,\cdot\rangle$ and let

$$A_0 : \mathscr{D}(A_0) \subseteq \mathscr{H} \to \mathscr{H}$$

a self-adjoint operator on it. We denote by $\mathscr{H}_{A_0}$ the Hilbert space given by the linear space $\mathscr{D}(A_0)$ endowed with the scalar product

$$\langle\phi,\psi\rangle_{A_0} = \langle\phi,\psi\rangle + \langle A_0\phi, A_0\psi\rangle.$$

Given then a Hilbert space $\mathfrak{h}$ with scalar product $(\cdot,\cdot)$ and a linear, bounded and surjective operator

$$\tau : \mathscr{H}_{A_0} \to \mathfrak{h},$$

such that $\mathscr{K}(\tau)$ is dense in $\mathscr{H}$; we denote by S the densely defined closed symmetric operator

$$S : \mathscr{K}(\tau) \subseteq \mathscr{H} \to \mathscr{H}, \qquad S\phi := A_0\phi.$$

Our aim is to provide, together with their resolvents, all self-adjoint extensions of S.

For any $z \in \rho(A_0)$ we define the bounded operators

$$\begin{aligned} &R_z := (-A_0 + z)^{-1} : \mathscr{H} \to \mathscr{H}_{A_0}, \\ &G_z := (\tau R_{\bar{z}})^* : \mathfrak{h} \to \mathscr{H}. \end{aligned} \tag{2.1}$$

By [17], lemma 2.1, given the surjectivity hypothesis $\mathscr{R}(\tau) = \mathfrak{h}$, the density assumption $\overline{\mathscr{K}(\tau)} = \mathscr{H}$ is equivalent to

$$\mathscr{R}(G_z) \cap \mathscr{D}(A_0) = \{0\}.$$

However, since by first resolvent identity,

$$(z-w)R_wG_z = G_w - G_z, \tag{2.2}$$

one has

$$\mathscr{R}(G_w - G_z) \subset \mathscr{D}(A_0).$$

From now on, even if this hypothesis can be avoided (see [16–19]), for the sake of simplicity we suppose that

$$0 \in \rho(A_0).$$

We define the family Γ_z, $z \in \rho(A_0)$, of bounded linear maps:

$$\Gamma_z : \mathfrak{h} \to \mathfrak{h}, \qquad \Gamma_z := \tau\,(G_0 - G_z) \equiv -z\tau A_0^{-1}G_z. \tag{2.3}$$

Given then an orthogonal projection,

$$\Pi : \mathfrak{h} \to \mathfrak{h}, \qquad \mathfrak{h}_0 \equiv \mathscr{R}(\Pi),$$

and a self-adjoint operator,

$$\Theta : \mathscr{D}(\Theta) \subseteq \mathfrak{h}_0 \to \mathfrak{h}_0,$$

we define the closed operator

$$\Gamma_{z,\Pi,\Theta} := (\Theta + \Pi\Gamma_z\Pi) : \mathscr{D}(\Theta) \subseteq \mathfrak{h}_0 \to \mathfrak{h}_0,$$

and the open set

$$Z_{\Pi,\Theta} := \{z \in \rho(A_0)\ :\ 0 \in \rho(\Gamma_{z,\Pi,\Theta})\}.$$

With such premises the following two theorems have straightforward proofs. Theorem 2.1 is an obvious modification (taking into account the hypothesis $0 \in \rho(A_0)$) of theorem 3.1 in [18] (also see [17], theorem 3.4); theorem 2.2 is the combination of theorem 2.1 with theorem 2.1 and theorem 2.4 in [19] (also see [16], theorem 2.1, [17], theorem 2.2, for the case $\Pi = 1$).

Theorem 2.1. *The adjoint of S is given by*

$$S^* : \mathscr{D}(S^*) \subseteq \mathscr{H} \to \mathscr{H}, \qquad S^*\phi = A_0\phi_0,$$

$$\mathscr{D}(S^*) = \{\phi \in \mathscr{H} : \phi = \phi_0 + G_0\zeta_\phi,\, \phi_0 \in \mathscr{D}(A_0),\, \zeta_\phi \in \mathfrak{h}\}.$$

Moreover,

$$\forall \phi, \psi \in \mathscr{D}(S^*), \quad \langle S^*\phi, \psi\rangle - \langle \phi, S^*\psi\rangle = (\tau\phi_0, \zeta_\psi) - (\zeta_\phi, \tau\psi_0). \tag{2.4}$$

Theorem 2.2. *The set $Z_{\Pi,\Theta}$ is not void,*

$$\mathbb{C}\backslash\mathbb{R} \subseteq Z_{\Pi,\Theta},$$

and

$$R_{z,\Pi,\Theta} := R_z + G_z\Pi\Gamma_{z,\Pi,\Theta}^{-1}\Pi G_{\bar z}^*, \qquad z \in Z_{\Pi,\Theta},$$

is the resolvent of the self-adjoint extension $A_{\Pi,\Theta}$ of S defined by

$$A_{\Pi,\Theta} : \mathscr{D}(A_{\Pi,\Theta}) \subseteq \mathscr{H} \to \mathscr{H}, \qquad A_{\Pi,\Theta}\phi = S^*\phi \equiv A_0\phi_0,$$

$$\mathscr{D}(A_{\Pi,\Theta}) = \{\phi \in \mathscr{D}(S^*) : \zeta_\phi \in \mathscr{D}(\Theta),\, \Pi\tau\phi_0 = \Theta\zeta_\phi\}.$$

Remark 2.3. Note that, since $\phi_0 = A_0^{-1}S^*\phi$,

$$\Pi\tau\phi_0 = \Theta\zeta_\phi \qquad \Longleftrightarrow \qquad \Pi\hat\tau_0\phi = \Theta\zeta_\phi,$$

where the regularized trace operator $\hat{\tau}_0$ is defined by

$$\hat{\tau}_0 : \mathscr{D}(S^*) \to \mathfrak{h}, \qquad \hat{\tau}_0\phi := \tau A_0^{-1} S^*\phi.$$

By exploiting the connection with von Neumann's theory (see [19], section 3; see also [17], section 4 for the case of relatively prime extensions) one obtains

Theorem 2.4. *The set of operators provided by theorem 2.2 coincides with the set $\mathscr{E}(S)$ of all self-adjoint extensions of the symmetric operator S. Thus $\mathscr{E}(S)$ is parametrized by the bundle $p : \mathsf{E}(\mathfrak{h}) \to \mathsf{P}(\mathfrak{h})$, where $\mathsf{P}(\mathfrak{h})$ denotes the set of orthogonal projections in $\mathfrak{h}$, and $p^{-1}(\Pi)$ is the set of self-adjoint operators in the Hilbert space $\mathscr{R}(\Pi)$. The set of self-adjoint operators in $\mathfrak{h}$, i.e. $p^{-1}(1)$, parametrizes all relatively prime extensions of S i.e. those for which $\mathscr{D}(\hat{A}) \cap \mathscr{D}(A_0) = \mathscr{D}(S)$.*

We conclude this section with a result about the spectral properties of the extensions (see [6], section 2, for point 1, and [18], theorem 3.4, for point 2).

Theorem 2.5.

(1)

$$\lambda \in \sigma_p(A_{\Pi,\Theta}) \cap \rho(A_0) \quad \iff \quad 0 \in \sigma_p(\Gamma_{\lambda,\Pi,\Theta}),$$

where $\sigma_p(\cdot)$ denotes the point spectrum. An analogous result holds for the continuous spectrum.

(2)

$$G_\lambda : \mathscr{K}(\Gamma_{\lambda,\Pi,\Theta}) \to \mathscr{K}(-A_{\Pi,\Theta} + \lambda)$$

is a bijection for any $\lambda \in \sigma_p(A_{\Pi,\Theta}) \cap \rho(A_0)$.

3. Extensions and Krein's Formula

Let $\Omega \subset \mathbb{R}^n$, $n > 1$, a bounded open set with a Lipschitz boundary. We denote by $H^k(\Omega)$ the Sobolev–Hilbert space given by the closure of $C^\infty(\bar{\Omega})$ with respect to the norm

$$\|u\|^2_{H^k(\Omega)} = \sum_{0\leqslant\alpha_1+\ldots+\alpha_n\leqslant k} \left\|\partial_1^{\alpha_1}\ldots\partial_n^{\alpha_n}u\right\|^2_{L^2(\Omega)}.$$

Analogously, $H^k_0(\Omega) \subsetneq H^k(\Omega)$ denotes the closure of $C^\infty_c(\Omega)$ with respect to the same norm.

Given the differential expression,

$$A = \nabla\cdot a\nabla - b\cdot\nabla - c \equiv \sum_{i,j=1}^n \partial_i(a_{ij}\partial_j) - \sum_{i=1}^n b_i\partial_i - c,$$

we suppose that the matrix $a(x) \equiv (a_{ij}(x))$ is Hermitean for a.e. $x \in \Omega$, that there exist $\mu_1 > 0$, $\mu_2 > 0$ such that

$$\forall\xi \in \mathbb{R}^n, \qquad \mu_1\|\xi\|^2 \leqslant \xi\cdot a(x)\xi \leqslant \mu_2\|\xi\|^2$$

and that

$$b_i \in L^q(\Omega), \qquad c \in L^{q/2}(\Omega), \qquad q = n \quad \text{if} \quad n \geqslant 3, \qquad q > 2 \quad \text{if} \quad n = 2.$$

Then A maps $H^1(\Omega)$ into $H^{-1}(\Omega)$ (see, e.g., [7], section 1, chapter VI), where $H^{-1}(\Omega)$ denotes the adjoint space of $H^1_0(\Omega)$, the sesquilinear form,

$$q_A : H^1_0(\Omega) \times H^1_0(\Omega) \to \mathbb{C}$$

$$q_A(u, v) := -\left(\langle \nabla u, a\nabla v\rangle_{L^2(\Omega)} + \langle u, b\cdot\nabla v\rangle_{L^2(\Omega)} + \langle u, cv\rangle_{L^2(\Omega)}\right),$$

is continuous and there exists a positive constant λ such that $-q_A + \lambda$ is coercive (see, e.g., [7], proposition 1.2, chapter VI). Thus by Lax–Milgram theorem (see, e.g., [7], theorem 1.4, chapter VI) there exists a unique closed, densely defined, linear operator,

$$A_0 : \mathscr{D}(A_0) \subseteq L^2(\Omega) \to L^2(\Omega), \qquad A_0 u = Au,$$

$$\mathscr{D}(A_0) = \left\{u \in H^1_0(\Omega) : Au \in L^2(\Omega)\right\},$$

such that

$$\forall u \in \mathscr{D}(A_0), \qquad \forall v \in H^1_0(\Omega), \qquad q_A(u, v) = \langle u, A_0 v\rangle_{L^2(\Omega)}.$$

Moreover, $\mathscr{D}(A_0)$ is dense in $H^1_0(\Omega)$; $0 \in \rho(-A_0 + \lambda)$; A_0 has a compact resolvent and its spectrum consists of an infinite sequence of eigenvalues λ_n, each having finite multiplicity and with $\mathrm{Re}\lambda_n < -\lambda$. An analogous result holds for the sesquilinear form q^*_A:

$$q^*_A(u, v) := \overline{q_A(v, u)},$$

and the operator corresponding to q^*_A is the adjoint A^*_0.

Suppose now that

$$\partial_i a_{ij} \in L^q(\Omega), \qquad q = n \qquad \text{if} \quad n \geqslant 3, \qquad q > 2 \qquad \text{if} \quad n = 2,$$

so that, by Sobolev embedding theorem, A is continuous from $H^2(\Omega)$ into $L^2(\Omega)$ and

$$H^2_0(\Omega) \subsetneq H^2(\Omega) \cap H^1_0(\Omega) \subseteq \mathscr{D}(A_0).$$

By interior regularity estimates (see, e.g., [13], section 7, chapter 3) $A{\restriction}_{C^\infty_c(\Omega)}$, the restriction of A to $C^\infty_c(\Omega)$, is closable and its closure is given by $A_{\min} \subsetneq A_0$, the minimal realization of A, defined by

$$A_{\min} : H^2_0(\Omega) \subseteq L^2(\Omega) \to L^2(\Omega), \qquad A_{\min} u := Au.$$

From now on we suppose that

$$q_A = q^*_A.$$

Thus A_0 is a self-adjoint operator, the Friedrichs extension of the closed symmetric operator $A_{\min}$, and one has

$$A^*_{\min} = (A{\restriction}_{C^\infty_c(\Omega)})^* = A_{\max},$$

where $A_{\max}$, the maximal realization of A, is defined by

$$A_{\max} : \mathscr{D}(A_{\max}) \subseteq L^2(\Omega) \to L^2(\Omega), \qquad A_{\max} u := Au,$$

$$\mathscr{D}(A_{\max}) := \{u \in L^2(\Omega) : Au \in L^2(\Omega)\}.$$

Hence,

$$\mathscr{D}(A_0) = H^1_0(\Omega) \cap \mathscr{D}(A_{\max}).$$

Moreover,

$$\mathscr{D}(A_{\min}) = H^2_0(\Omega) \subsetneq \mathscr{D}(A_{\max}),$$

so that $A{\restriction}_{C^\infty_c(\Omega)}$ is not essentially self-adjoint:

$$A_{\min} \subsetneq A_0 \subsetneq A_{\max},$$

and the symmetric operator $A_{\min}$ has infinitely many self-adjoint extensions. We want now to find all such extensions and to give their resolvents. In order to render straightforward the application of the results given in section 2, we would like to have a more explicit characterization of $\mathscr{D}(A_0)$. Thus in the following we impose more stringent hypotheses on the set Ω.

Suppose that the boundary of Ω is a piecewise C^2 surface with curvature bounded from above and that $a_{ij} \in C(\bar{\Omega})$ when $n \geqslant 3$. Then, by global regularity results (see, e.g., [13], chapter 3, section 11), the graph norm of $A_{\max}$ is equivalent to that of $H^2(\Omega)$ on $C_0^\infty(\bar{\Omega})$, the space of smooth functions on Ω which vanish on its boundary $\partial\Omega$. Thus $A\restriction_{C_0^\infty(\bar{\Omega})}$, the restriction of A to $C_0^\infty(\bar{\Omega})$, is closable and its closure is given by

$$\tilde{A}_0 : \tilde{H}_0^2(\Omega) \subseteq L^2(\Omega) \to L^2(\Omega), \qquad \tilde{A}_0 u := Au,$$

where $\tilde{H}_0^2(\Omega)$ denotes the closure of $C_0^\infty(\bar{\Omega})$ with respect to the $H^2(\Omega)$ norm.

Without further hypotheses on Ω, $\tilde{A}_0 \neq A_0$ is possible: for example, if Ω is a non-convex plane polygon then the Laplace operator Δ is not self-adjoint on $\tilde{H}_0^2(\Omega)$. Indeed, by [2] it has deficiency indices $(d_-, d_+) = (d, d)$, where d is the number of non-convex corners .

Suppose now that the a_{ij}'s are Lipschitz continuous up to the boundary and that $\partial\Omega$ is $C^{1,1}$, i.e. it is locally the graph of a C^1 function with Lipschitz derivatives (see, e.g., [11], section 1.2, for the precise definition). Then (see, e.g., [14], chapter 1, section 8.2, [11], section 1.5) there are unique continuous and surjective linear maps:

$$\rho : H^1(\Omega) \to H^{1/2}(\partial\Omega),$$

$$\gamma_a : H^2(\Omega) \to H^{3/2}(\partial\Omega) \oplus H^{1/2}(\partial\Omega), \qquad \gamma_a\phi := (\rho\phi, \tau_a\phi)\,,$$

such that

$$\rho\phi(x) := \phi(x), \qquad \tau_a\phi(x) \equiv \frac{\partial\phi}{\partial\nu_a}(x) := \sum_{i,j=1}^{n} a_{ij}(x)\nu_i(x)\partial_j\phi(x)$$

for any $\phi \in C^\infty(\bar{\Omega})$ and $x \in \partial\Omega$. Here $\nu \equiv (\nu_1, \dots, \nu_n)$ denotes the outward normal vector on $\partial\Omega$ and $H^s(\partial\Omega)$, $s > 0$, are the usual fractional Sobolev–Hilbert spaces on $\partial\Omega$ (see, e.g., [11], section 1.3.3). Moreover, Green's formula holds: for any $u \in H^2(\Omega)$ and $v \in H^2(\Omega) \cap H_0^1(\Omega)$ one has

$$\langle Au, v\rangle_{L^2(\Omega)} = \langle u, A_0 v\rangle_{L^2(\Omega)} - \langle \rho u, \tau_a v\rangle_{L^2(\partial\Omega)}. \tag{3.1}$$

By proceeding as in the proof of theorem 6.5 in [14][chapter 6] (which uses (3.1)) the map γ_a can be extended to (see the comment in [11] before theorem 1.5.3.4)

$$\hat{\gamma}_a : \mathscr{D}(A_{\max}) \to H^{-1/2}(\partial\Omega) \oplus H^{-3/2}(\partial\Omega), \qquad \hat{\gamma}_a\phi = (\hat{\rho}\phi, \hat{\tau}_a\phi),$$

where $H^{-s}(\partial\Omega)$ denotes the adjoint space of $H^s(\partial\Omega)$, and Green's formula (3.1) can be extended to the case in which $u \in \mathscr{D}(A_{\max})$:

$$\langle A_{\max}u, v\rangle_{L^2(\Omega)} = \langle u, A_0 v\rangle_{L^2(\Omega)} - (\hat{\rho}u, \tau_a v)_{-\frac{1}{2},\frac{1}{2}}. \tag{3.2}$$

Here $(\cdot,\cdot)_{-\frac{1}{2},\frac{1}{2}}$ denotes the duality between $H^{1/2}(\partial\Omega)$ and $H^{-1/2}(\partial\Omega)$. With such definitions of ρ and τ one has (see, e.g., [11], corollary 1.5.1.6)

$$H_0^1(\Omega) = H^1(\Omega) \cap \mathscr{K}(\rho), \qquad H_0^2(\Omega) = H^2(\Omega) \cap \mathscr{K}(\gamma_1).$$

Moreover, by the stated properties of ρ and $\hat{\rho}$, by the equivalence of the graph norm of $A_{\max}$ with the $H^2(\Omega)$ norm on $\tilde{H}_0^2(\Omega)$ and by the density of $C^\infty(\bar{\Omega})$ in $\mathscr{D}(A_{\max})$, one gets the equalities

$$\tilde{H}_0^2(\Omega) = H^2(\Omega) \cap H_0^1(\Omega) = \mathscr{D}(A_{\max}) \cap H_0^1(\Omega) \equiv \mathscr{D}(A_0),$$

so that $\tilde{A}_0 = A_0$.

In conclusion, we can apply the results given in section 2 (by adding, if necessary, a constant to A_0 we may suppose that $0 \in \rho(A_0)$) to the self-adjoint operator

$$A_0 : H^2(\Omega) \cap H^1_0(\Omega) \subseteq L^2(\Omega) \to L^2(\Omega), \qquad A_0 u := Au,$$

with $S = A_{\min}$, $\mathfrak{h} = H^{1/2}(\partial\Omega)$ and

$$\tau : H^2(\Omega) \cap H^1_0(\Omega) \to H^{1/2}(\partial\Omega), \qquad \tau := \tau_a.$$

Note that $\mathscr{K}(\tau) = H^2_0(\Omega)$ since $\mathscr{K}(\gamma_a) = \mathscr{K}(\gamma_1)$ by $\nu(x)\cdot a(x)\nu(x) \geqslant \mu_1 > 0$, $x \in \partial\Omega$, and that τ is surjective by the surjectivity of γ_a.

Thus, by theorem 2.4, under the hypotheses above, the set $\mathscr{E}(A_{\min})$ of all self-adjoint extensions of $A_{\min}$ can be parametrized by the bundle

$$p : \mathsf{E}(H^{1/2}(\partial\Omega)) \to \mathsf{P}(H^{1/2}(\partial\Omega)).$$

Now, in order to write down the extensions of $A_{\min}$ together with their resolvents, we make explicit the operator G_z defined in (2.1). By theorem 2.1, since $A_{\max} = A^*_{\min}$, we have

$$\mathscr{D}(A_{\max}) = \{u = u_0 + G_0 h, u_0 \in H^2(\Omega) \cap H^1_0(\Omega), h \in H^{1/2}(\partial\Omega)\},$$

$$A_{\max} u = A_0 u_0.$$

Thus $A_{\max} G_0 h = 0$ and so by (3.2) there follows, for all $h \in H^{1/2}(\partial\Omega)$ and for all $u \in \mathscr{D}(A_0)$,

$$\langle G_0 h, A_0 u\rangle_{L^2(\Omega)} = (\hat{\rho} G_0 h, \tau_a u)_{-\frac{1}{2},\frac{1}{2}}.$$

Since, by (2.4),

$$\langle G_0 h, A_0 u\rangle_{L^2(\Omega)} = \langle G_0 h, A_{\max} u\rangle_{L^2(\Omega)} = \langle G_0 h, A^*_{\min} u\rangle_{L^2(\Omega)} = \langle h, \tau_a u\rangle_{H^{1/2}(\partial\Omega)},$$

one obtains $\hat{\rho} G_0 h = \Lambda h$, where

$$\Lambda : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$$

is the unitary operator defined by

$$\forall h_1, h_2 \in H^{1/2}(\partial\Omega), \qquad (\Lambda h_1, h_2)_{-\frac{1}{2},\frac{1}{2}} = \langle h_1, h_2\rangle_{H^{1/2}(\partial\Omega)}.$$

For successive notational convenience we pose $\Sigma := \Lambda^{-1}$.

Remark 3.1. If $\partial\Omega$ carries a Riemannian structure then $H^s(\partial\Omega)$ can be defined as the completion of $C^\infty(\partial\Omega)$ with respect to the scalar product

$$\langle f, g\rangle_{H^s(\partial\Omega)} := \langle f, (-\Delta_{\mathrm{LB}} + 1)^s g\rangle_{L^2(\partial\Omega)}.$$

Here the self-adjoint operator Δ_{LB} is the Laplace–Beltrami operator in $L^2(\partial\Omega)$. With such a definition $(-\Delta_{\mathrm{LB}} + 1)^{1/2}$ can be extended to the unitary map Λ.

Since $G_z = G_0 + zA_0^{-1} G_z$ by (2.2), $G_z h$ is the solution of the Dirichlet boundary value problem:

$$\begin{cases} A_{\max} G_z h = z G_z h, \\ \hat{\rho} G_z h = \Lambda h. \end{cases} \tag{3.3}$$

Thus we can write $G_0 \Sigma = K$, where $K : H^{-1/2}(\partial\Omega) \to \mathscr{D}(A_{\max})$ is the Poisson operator which provides the solution of the Dirichlet problem with boundary data in $H^{-1/2}(\partial\Omega)$. Analogously we define $K_z : H^{-1/2}(\partial\Omega) \to \mathscr{D}(A_{\max})$ by $K_z := G_z \Sigma$. Note that $G_0 h$, hence $G_z h$, is uniquely defined as the solution of (3.3): for any other solution u one has $u - G_0 h \in \mathscr{K}(A_0) = \{0\}$.

Now, according to (2.3), we define the bounded linear operator

$$\Gamma_z : H^{1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega), \qquad \Gamma_z := \tau(G_0 - G_z),$$

which, by (2.2) and the definitions of K and K_z, can be re-written as

$$\Gamma_z = -z\tau_a A_0^{-1} G_z \equiv z\tau_a R_z K \Lambda \equiv (\hat{\tau}_a K - \hat{\tau}_a K_z)\Lambda. \tag{3.4}$$

By $\hat{\rho} G_0 h = \Lambda h$, by theorem 2.1 and remark 2.3, we can define the regularized trace operator

$$\begin{aligned}&\hat{\tau}_{a,0} : \mathscr{D}(A_{\max}) \to H^{1/2}(\partial\Omega),\\ &\hat{\tau}_{a,0} u := \tau_a(u - G_0 \Sigma \hat{\rho} u) \equiv \hat{\tau}_a u - P_a \hat{\rho} u \equiv \tau_a A_0^{-1} A_{\max} u,\end{aligned} \tag{3.5}$$

where the linear operator P_a, known as the Dirichlet-to-Neumann operator over $\partial\Omega$, is defined by

$$P_a : H^{-1/2}(\partial\Omega) \to H^{-3/2}(\partial\Omega), \qquad P_a := \hat{\tau}_a K.$$

In conclusion, by theorems 2.2 and 2.4, one has the following

Theorem 3.2. *Any self-adjoint extension $\hat{A}$ of $A_{\min}$ is of the kind*

$$\hat{A} : \mathscr{D}(\hat{A}) \subseteq L^2(\Omega) \to L^2(\Omega), \qquad \hat{A} u = A_{\max} u,$$

$$\mathscr{D}(\hat{A}) = \{u \in \mathscr{D}(A_{\max}) : \Sigma\hat{\rho} u \in \mathscr{D}(\Theta), \quad \Pi \hat{\tau}_{a,0} u = \Theta \Sigma \hat{\rho} u\},$$

where $(\Pi, \Theta) \in \mathsf{E}(H^{1/2}(\partial\Omega))$, and

$$(-\hat{A} + z)^{-1} = (-A_0 + z)^{-1} + G_z \Pi (\Theta + \Pi \Gamma_z \Pi)^{-1} \Pi G_{\bar{z}}^*,$$

with $\tau_{a,0}$, G_z and Γ_z defined by (3.5), (3.3) and (3.4), respectively.

Remark 3.3. When the boundary is smooth, by proceeding as in [19], example 5.5, in the case the $L^2(\partial\Omega)$-symmetric, bounded linear operator $B : H^{3/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)$ is such that $\Theta_B := (-P_a + B)\Lambda$, $\mathscr{D}(\Theta_B) = H^{5/2}(\partial\Omega)$, is self-adjoint ($B$ pseudo-differential of order strictly less than one suffices); the extension A_B corresponding to $(1, \Theta_B)$ has the domain defined by Robin-type boundary conditions:

$$\mathscr{D}(A_B) := \{u \in H^2(\Omega) : \tau_a u = B\rho\}.$$

4. A simple example

One of the simplest examples is given by a rotation-invariant second-order elliptic differential operator on the unit disc $D \subset \mathbb{R}^2$. Thus we consider the self-adjoint extensions of

$$A_{\min} : H_0^2(D) \subset L^2(D) \to L^2(D), \qquad A_{\min} u = Au,$$

where

$$A = \nabla \cdot a \nabla - c, \qquad a_{ij}(x) = a(\|x\|)\delta_{ij}, \qquad c(x) = c(\|x\|).$$

We suppose that a is Lipschitz continuous, $\inf_{0 \leqslant r \leqslant 1} a(r) > 0$, and that $c \in L^q((0, 1); r\,\mathrm{d}r)$, $q > 2$. By adding, if necessary, a constant to c we suppose that $-A_0 > 0$.

In $L^2(D) \simeq L^2((0, 1); r\,\mathrm{d}r) \otimes L^2((0, 2\pi); \mathrm{d}\varphi)$ we use the orthonormal basis $\{U_{mn}\}$, $m \in \mathbb{N}$, $n \in \mathbb{Z}$,

$$U_{mn}(r, \varphi) = u_{m|n|}(r) \frac{\mathrm{e}^{in\varphi}}{\sqrt{2\pi}}$$

made by the normalized eigenfunctions of the Friedrichs extension A_0 of A. Here $\{u_{mn}\}$, $m \in \mathbb{N}$, is the orthonormal basis in $L^2((0, 1); r\,\mathrm{d}r)$ made by the normalized eigenfunctions of the self-adjoint Sturm–Liouville operator:

$$L_n f(r) = -\frac{1}{r}(ra(r) f'(r))' + \left(c(r) + \frac{n^2}{r^2}\right) f(r), \qquad n \geqslant 0,$$

with boundary conditions $f(1_-)=0$ and $\lim_{r\to 0_+} rf'(r)=0$ if $n=0$, $f(0_+)=0$ if $n\neq 0$. Denoting by $\lambda_{mn}^2>0$, $m\in\mathbb{N}$, the eigenvalues of L_n, one has

$$\sigma(A_0)=\sigma_d(A_0)=\left\{-\lambda_{m|n|}^2,\, m\in\mathbb{N},\, n\in\mathbb{Z}\right\}.$$

In $H^{1/2}(S^1)$ we use the orthonormal basis $\{e_k\}$, $k\in\mathbb{Z}$, defined by

$$e_k(\varphi):=\frac{\mathrm{e}^{ik\varphi}}{\sqrt{2\pi}(k^2+1)^{1/4}}.$$

We want now to compute the matrix elements, relative to the basis $\{U_{mn}\}$, of the resolvents of the self-adjoint extensions of $A_{\min}$.

By defining

$$\nu_{mn}:=\lim_{r\uparrow 1} a(r)u'_{mn}(r),$$

one has

$$\begin{aligned}[G_0]_{mnk} &:= \langle U_{mn}, G_0 e_k\rangle_{L^2(D)} = \langle G_0^* U_{mn}, e_k\rangle_{H^{1/2}(S^1)} =: \overline{[G_0^*]_{kmn}}\\ &= \langle \tau_a(-A_0)^{-1}U_{mn}, e_k\rangle_{H^{1/2}(S^1)} = (n^2+1)^{1/4}\frac{\nu_{m|n|}}{\lambda_{m|n|}^2}\delta_{nk}.\end{aligned}$$

Since $G_z=G_0-z(-A_0+z)^{-1}G_0$, one has then

$$\begin{aligned}[G_z]_{mnk} &= \overline{[G_z^*]_{kmn}} = [G_0]_{mnk} - \frac{z}{\lambda_{m|n|}^2+z}[G_0]_{mnk}\\ &= \frac{\lambda_{m|n|}^2}{\lambda_{m|n|}^2+z}[G_0]_{mnk} = (n^2+1)^{1/4}\frac{\nu_{m|n|}}{\lambda_{m|n|}^2+z}\delta_{nk}.\end{aligned}$$

Analogously

$$\begin{aligned}[\Gamma_z]_{ik} &:= -z\langle e_i, \tau_a(-A_0+z)^{-1}G_0 e_k\rangle_{H^{1/2}(S^1)}\\ &= -z(k^2+1)^{1/2}\sum_{m=1}^{\infty}\frac{\nu_{m|k|}^2}{\lambda_{m|k|}^2\left(\lambda_{m|k|}^2+z\right)}\delta_{ik}.\end{aligned}$$

Thus, in the case the orthogonal projection Π is the one corresponding to the subspace of $H^{1/2}(S^1)$ generated by $\{e_k, k\in I\}$, $I\subseteq\mathbb{Z}$, and $[\Theta]_{ik}=\theta_k\delta_{ik}$, $k\in I$, by theorem 2.2 one obtains

$$\begin{aligned}[(-A_{\Pi,\Theta}+z)^{-1}]_{mn\tilde m\tilde n} &:= \langle U_{mn}, (-A_{\Pi,\Theta}+z)^{-1}U_{\tilde m\tilde n}\rangle_{L^2(D)}\\ &= \frac{\delta_{m\tilde m}\delta_{n\tilde n}}{\lambda_{m|n|}^2+z} + \frac{(n^2+1)^{1/2}}{\theta_n+[\Gamma_z]_{nn}}\frac{\nu_{m|n|}}{\lambda_{m|n|}^2+z}\frac{\nu_{\tilde m|n|}}{\lambda_{\tilde m|n|}^2+z}\delta_{n\tilde n}\end{aligned}$$

for any $n\in I$, and

$$[(-A_{\Pi,\Theta}+z)^{-1}]_{mn\tilde m\tilde n} = \frac{\delta_{m\tilde m}\delta_{n\tilde n}}{\lambda_{m|n|}^2+z}$$

for any $n\notin I$. Once the resolvent has been written as above, by theorem 2.5 given any sequence

$$\{\lambda_n\}_{n\in I}\subset\mathbb{R}\cap\rho(A_0),$$

posing

$$\theta_n:=-[\Gamma_{\lambda_n}]_{nn}, \qquad n\in I,$$

one obtains

$$\{\lambda_n\}_{n\in I}\subset\sigma_p(A_{\Pi,\Theta}).$$

Moreover,

$$U_n = \sum_{m=1}^{\infty} \frac{\nu_{m|n|}}{\lambda_{m|n|}^2 + \lambda_n} U_{mn}$$

is the eigenfunction with eigenvalue λ_n.

Remark 4.1. The previous example can be re-phrased in the language of decomposable operators (see, e.g., [21], section 13.16): the operator A_0 is decomposable with fibers $A_0(n) = -L_{|n|}$, and the decomposable self-adjoint extensions of $A_{\min}$ have decomposable resolvents with fibers given by the resolvents of the self-adjoint extensions of the fibers $A_{\min}(n)$, which are symmetric operators with deficiency indices $(1, 1)$. However this approach gives a less (than that provided by theorem 3.2) explicit expression for the self-adjointness domain.

Remark 4.2. In the case $a = 1, c = 0$, one has

$$\lambda_{mn} = \mu_{mn}, \qquad u_{mn}(r) = c_{mn} J_n(\mu_{mn} r),$$

where J_n denotes the nth order Bessel function, μ_{mn} is its mth positive zero, and c_{mn} is the normalization constant. Thus

$$\nu_{mn} = -c_{mn}\mu_{mn} J_{n+1}(\mu_{mn}).$$

The following remark shows that the boundary conditions corresponding to couples (Π, Θ) of the above kind can be quite different from the usual ones.

Remark 4.3. Suppose in the previous example we take $a = 1, c = 0$, i.e. $A = \Delta$ and $I = \{0\}$, $\lambda_0 = 0$. Then

$$\Pi : H^{1/2}(S^1) \to \mathbb{C}, \qquad \Pi f = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(\varphi)\, \mathrm{d}\varphi,$$

and $\Theta : \mathbb{C} \to \mathbb{C}$ is the multiplication by zero, since $\Gamma_0 = 0$. Thus

$$\mathscr{D}(\Delta_{\Pi,0}) = \left\{ u \in \mathscr{D}(\Delta_{\max}) : \Sigma\hat{\rho} u = \text{const}, \int_0^{2\pi} \hat{\tau}_{1,0} u(\varphi)\, \mathrm{d}\varphi = 0 \right\}.$$

Since $\Lambda \equiv \Sigma^{-1}$ maps constants into constants,

$$\{u \in \mathscr{D}(\Delta_{\max}) : \hat{\rho} u = \text{const}\} = \{u \in H^2(D) : \rho u = \text{const}\}$$

by elliptic regularity, and

$$\int_0^{2\pi} \hat{\tau}_{1,0} u(\varphi)\, \mathrm{d}\varphi = \int_0^{2\pi} [\tau_1 \Delta_0^{-1} \Delta_{\max} u](\varphi)\, \mathrm{d}\varphi = \int_0^{2\pi} \tau_1 u(\varphi)\, \mathrm{d}\varphi,$$

in conclusion one has

$$\mathscr{D}(\Delta_{\Pi,0}) = \left\{ u \in H^2(D) : \rho u = \text{const}, \int_0^{2\pi} \rho \frac{\partial u}{\partial r}(\varphi)\, \mathrm{d}\varphi = 0 \right\}.$$

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